

SELF-ADJOINT INDEFINITE LAPLACIANS

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ABSTRACT. Let Ω_- and Ω_+ be two bounded smooth domains in \mathbb{R}^n , $n \geq 2$, separated by a hypersurface Σ . For $\mu > 0$, consider the function $h_\mu = 1_{\Omega_-} - \mu 1_{\Omega_+}$. We discuss self-adjoint realizations of the operator $L_\mu = -\nabla \cdot h_\mu \nabla$ in $L^2(\Omega_- \cup \Omega_+)$ with the Dirichlet condition at the exterior boundary. We show that L_μ is always essentially self-adjoint on the natural domain (corresponding to transmission-type boundary conditions at the interface Σ) and study some properties of its unique self-adjoint extension $\mathcal{L}_\mu := \overline{L_\mu}$. If $\mu \neq 1$, then \mathcal{L}_μ simply coincides with L_μ and has compact resolvent. If $n = 2$, then \mathcal{L}_1 has a non-empty essential spectrum, $\sigma_{\text{ess}}(\mathcal{L}_1) = \{0\}$. If $n \geq 3$, the spectral properties of \mathcal{L}_1 depend on the geometry of Σ . In particular, it has compact resolvent if Σ is the union of disjoint strictly convex hypersurfaces, but can have a non-empty essential spectrum if a part of Σ is flat. Our construction features the method of boundary triplets, and the problem is reduced to finding the self-adjoint extensions of a pseudodifferential operator on Σ . We discuss some links between the resulting self-adjoint operator \mathcal{L}_μ and some effects observed in negative-index materials.

1. INTRODUCTION

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be an open bounded set with a smooth boundary $\partial\Omega$. Let Ω_- be a subset of Ω having a smooth boundary Σ (called *interface*) and such that $\overline{\Omega_-} \subset \Omega$. In addition, we consider the open set $\Omega_+ := \Omega \setminus \overline{\Omega_-}$, whose boundary is $\partial\Omega_+ = \Sigma \cup \partial\Omega$, and denote by N_\pm the unit normal on Σ exterior with respect to Ω_\pm . For $\mu > 0$, consider the function $h : \Omega \setminus \Sigma \rightarrow \mathbb{R}$,

$$h_\mu(x) = \begin{cases} 1, & x \in \Omega_-, \\ -\mu, & x \in \Omega_+. \end{cases}$$

The aim of the present work is to construct self-adjoint operators in $L^2(\Omega)$ corresponding to the formally symmetric differential expression $L_\mu = -\nabla \cdot h_\mu \nabla$. The operators of such a type appear e.g. in the study of negative-index metamaterials, and we refer to the recent paper [29] for a survey and an extensive bibliography; we remark that the parameter μ is usually called *contrast*. A possible approach is to consider the sesquilinear form

$$\ell_\mu(u, v) = \int_{\Omega} h_\mu \overline{\nabla u} \cdot \nabla v \, dx, \quad u, v \in H_0^1(\Omega),$$

and then to define L_μ as the operator generated by ℓ_μ , in particular, for all functions v from the domain of L_μ one should then have

$$\int_{\Omega} \overline{u} L_\mu v \, dx = \ell_\mu(u, v), \quad u \in H_0^1(\Omega). \quad (1)$$

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But the form ℓ_μ is not semibounded below, hence, the operator obtained in this way can have exotic properties, in particular, its self-adjointness is not guaranteed. We refer to [19, 35, 36] for some available results in this direction.

In [4] a self-adjoint operator for the above expression was constructed for a very particular geometry when $\Omega_- = (-1, 0) \times (0, 1)$ and $\Omega_+ = (0, 1) \times (0, 1)$, which enjoys a separation of variables and some symmetries. An interesting feature of the model is the possibility of a non-empty essential spectrum although the domain is bounded. Constructing self-adjoint operators realizations of L_μ for the general case is an open problem, see [23]. In the present note, we give a solution in the case of a smooth interface.

One should remark that the study of various boundary value problems involving differential expressions $\nabla \cdot h \nabla$ with sign-changing h has a long history, and the most classical form involves unbounded domains with a suitable radiation condition at infinity, cf. [12, 17, 31]. In particular, the geometric conditions appearing in the main results below are very close to those of [28, 31] for the well-posedness of a transmission problem. The case of a non-smooth interface Σ , which was partially studied in [7, 8, 9], is not covered by our approach.

In fact, the problem of self-adjoint realizations the non-critical case $\mu \neq 1$ was essentially settled in [7], while for the critical case $\mu = 1$ was only studied for the above-mentioned example of [4], in [36, Chapter 8] for another similar situation (symmetric Ω_- and Ω_+ separated by a finite portion of a hyperplane), and in [20] for the one-dimensional case. In a sense, in the present work we recast some techniques of the transmission problems and the pseudodifferential operators into the setting of self-adjoint extensions. Using the machinery of boundary triplets we reduce the problem first to finding self-adjoint extensions of a symmetric differential operator and then to the analysis of the associated Weyl function acting on the interface Σ . Then one arrives at the study of the essential self-adjointness of a pseudodifferential operator on Σ , whose properties depend on the dimension. We hope that, in view of the recent progress in the theory of self-adjoint extensions of partial differential operators, see e.g. [5, 6, 15], such a direct reformulation could be a starting point for a further advance in the study of indefinite operators.

Similar to [4], our approach is based on the theory of self-adjoint extensions. Using the natural identification $L^2(\Omega) \simeq L^2(\Omega_-) \oplus L^2(\Omega_+)$, $u \simeq (u_-, u_+)$, we introduce the sets

$$\mathcal{D}_\mu^s(\Omega \setminus \Sigma) := \left\{ u = (u_-, u_+) \in H^s(\Omega_-) \oplus H^s(\Omega_+) : \Delta u_\pm \in L^2(\Omega_\pm), \right. \\ \left. u_- = u_+ \text{ and } N_- \cdot \nabla u_- = \mu N_+ \cdot \nabla u_+ \text{ on } \Sigma, \quad u_+ = 0 \text{ on } \partial\Omega \right\}, \quad s \geq 0.$$

Here and below, the values at the boundary are understood as suitable Sobolev traces; the exact definitions are given in Section 3. Let us recall that for $\frac{1}{2} < s < \frac{3}{2}$ and $u = (u_-, u_+) \in H^s(\Omega_-) \oplus H^s(\Omega_+)$ the conditions $u_- = u_+$ on Σ and $u_+ = 0$ on $\partial\Omega$ entail $u \in H_0^s(\Omega)$, see e.g. [1, Theorem 3.5.1]. In particular,

$$\mathcal{D}_\mu^2(\Omega \setminus \Sigma) \subseteq H_0^{\frac{3}{2}-\varepsilon}(\Omega) \text{ for } \varepsilon > 0, \quad \mathcal{D}_\mu^1(\Omega \setminus \Sigma) \subseteq H_0^1(\Omega). \quad (2)$$

Consider the operator

$$L_\mu(u_-, u_+) = (-\Delta u_-, \mu \Delta u_+), \text{ with } \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \quad (3)$$

acting on the domain

$$\text{dom } L_\mu = \mathcal{D}_\mu^2(\Omega \setminus \Sigma). \quad (4)$$

Remark that L_μ satisfies (1) and it is a densely defined symmetric operator in $L^2(\Omega)$. Therefore, we use L_μ as a starting point and seek its self-adjoint extensions. Even if the case $\mu \neq 1$ was studied earlier, we include it into consideration as it does not imply any additional difficulties.

Theorem 1 (Self-adjointness). *The operator L_μ is essentially self-adjoint, and we denote*

$$\mathcal{L}_\mu := \overline{L_\mu}$$

its closure and unique self-adjoint extension. Furthermore, if $\mu \neq 1$, then $\mathcal{L}_\mu = L_\mu$, i.e. L_μ itself is self-adjoint, and has compact resolvent.

Now we consider in greater detail the critical case $\mu = 1$. The properties of \mathcal{L}_1 appear to depend on the dimension. In two dimensions, we have a complete result:

Theorem 2 (Critical contrast in two dimensions). *Let $\mu = 1$ and $n = 2$, then*

$$\text{dom } \mathcal{L}_1 = \mathcal{D}_1^0(\Omega \setminus \Sigma), \quad \mathcal{L}_1(u_-, u_+) = (-\Delta u_-, \Delta u_+), \quad (5)$$

and the essential spectrum is non-empty, $\sigma_{\text{ess}}(\mathcal{L}_1) = \{0\}$.

Remark (see Proposition 4 below) that 0 is not necessarily an eigenvalue of \mathcal{L}_1 , contrary to the preceding examples given in [4] and [36, Chapter 8] for which the essential spectrum consisted of an infinitely degenerate zero eigenvalue.

In dimensions $n \geq 3$ the result appears to depend on the geometric properties of Σ :

Theorem 3 (Critical contrast in dimensions ≥ 3). *Let $\mu = 1$ and $n \geq 3$, then \mathcal{L}_1 acts as $\mathcal{L}_1(u_-, u_+) = (-\Delta u_-, \Delta u_+)$, and its domain satisfies*

$$\mathcal{D}_1^1(\Omega \setminus \Sigma) \subseteq \text{dom } \mathcal{L}_1. \quad (6)$$

Furthermore,

- (a) *If on each connected component of Σ the principal curvatures are either all strictly positive or all strictly negative (in particular, if each maximal connected component of Σ is strictly convex), then*

$$\text{dom } \mathcal{L}_1 = \mathcal{D}_1^1(\Omega \setminus \Sigma) \quad (7)$$

and \mathcal{L}_1 has compact resolvent.

- (b) *If a subset of Σ is isometric to a non-empty open subset of \mathbb{R}^{n-1} , then*

$$\text{dom } \mathcal{L}_1 \neq \mathcal{D}_1^s(\Omega \setminus \Sigma) \quad \text{for any } s > 0, \quad (8)$$

the essential spectrum of \mathcal{L}_1 is non-empty, and $\{0\} \subseteq \sigma_{\text{ess}}(\mathcal{L}_1)$.

The proofs of the three theorems are given in Sections 2–4. In section 2 we recall the tools of the machinery of boundary triplets for self-adjoint extensions of symmetric operators. In section 3 we apply these tools to the operators L_μ and reduce the initial problem to finding self-adjoint extensions of a pseudodifferential operator Θ_μ acting on Σ , which is essentially a linear combination of (suitably defined) Dirichlet-to-Neumann maps on Ω_\pm . The self-adjoint extensions of Θ_μ are studied in Section 4 using a combination of some facts about Dirichlet-to-Neumann maps and pseudodifferential operators.

In addition, we use the definition of the operators \mathcal{L}_μ to revisit some results concerning the so-called cloaking by negative materials, see e.g [29, 30]. For $\delta > 0$, consider the operator

$T_{\mu,\delta}$ generated by the regularized sesquilinear form

$$t_{\mu,\delta}(u, v) := \int_{\Omega \setminus \Sigma} \overline{\nabla u} \cdot (h_\mu + i\delta) \nabla v \, dx, \quad u, v \in H_0^1(\Omega).$$

By the Lax-Milgram theorem, the operator $T_{\mu,\delta} : L^2(\Omega) \supset H_0^1(\Omega) \supset \text{dom } T_{\mu,\delta} \rightarrow L^2(\Omega)$ has a bounded inverse, hence, for $g \in L^2(\Omega)$ one may define $u_\delta := (T_{\mu,\delta})^{-1}g \in H_0^1(\Omega)$. It was observed in [30] that the limit properties of u_δ as δ tends to 0 can be quite irregular, in particular, the norm $\|u_\delta\|_{H^1(V)}$ may remain bounded for some subset $V \subset \Omega$ while $\|u_\delta\|_{H^1(\Omega \setminus V)}$ goes to infinity. The most prominent example is as follows: for $0 < r < R$ we denote

$$B_r := \{x \in \mathbb{R}^n : |x| < r\}, \quad B_{r,R} := \{x \in \mathbb{R}^n : r < |x| < R\}, \quad S_r := \{x \in \mathbb{R}^n : |x| = r\},$$

pick three constants $0 < r_i < r_e < R$ and consider the above operator $T_{\mu,\delta}$ corresponding to

$$\Omega := B_R, \quad \Omega_- := B_{r_i, r_e}, \tag{9}$$

and set $u_\delta := (T_{\mu,\delta})^{-1}g$ with g supported in $B_{r_e, R}$. Then the norm $\|u_\delta\|_{H^1(\Omega)}$ remains bounded for δ approaching 0 provided $\mu \neq 1$ or $n \geq 3$. For $\mu = 1$ and $n = 2$ the situation appears to be different: if g is supported outside the ball B_a with $a = r_e^2/r_i$, then $\|u_\delta\|_{H^1(B_R)}$ remains bounded, otherwise, for a generic g , the norm $\|u_\delta\|_{H^1(B_{r_e, R})}$ is bounded, while $\|u_\delta\|_{H^1(B_{r_i, r_e})}$ becomes infinite, see [30]. Such a non-uniform blow-up the H^1 norm is often referred to as an anomalously localized resonance, and we refer to [2, 3, 10, 22, 27] for a discussion of other similar models and generalizations.

It is interesting to understand whether similar observations can be made based on the direct study of the operator \mathcal{L}_μ . In fact, instead of taking the limit of u_δ one may study directly the solutions u of $\mathcal{L}_\mu u = g$. If $\mu \neq 1$, then $u \in H_0^1(\Omega)$ by Theorem 1. Furthermore, due to Theorem 3(b) the same holds for $\mu = 1$ and $n \geq 3$ as the interface Σ consists of two strictly convex hypersurfaces (the spheres S_{r_i} and S_{r_e}), which is quite close to the discussion of [21]; we remark that a separation of variables shows that \mathcal{L}_1 is injective and thus surjective in this case. The study of the case $\mu = 1$ and $n = 2$ is more involved, and the link to the anomalously localized resonance appears as follows (we assume a special form of the function g to make the discussion more transparent):

Proposition 4. *Let $\mu = 1$ and $n = 2$, then the operator \mathcal{L}_1 associated with (9) is injective, and a function $g \in L^2(B_R)$ of the form*

$$g(r \cos \theta, r \sin \theta) = 1_{(a,b)}(r)h(\theta), \quad h \in L^2(0, 2\pi), \quad r_e \leq a < b \leq R, \tag{10}$$

belongs to $\text{ran } \mathcal{L}_1$ if and only if

$$\sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{|h_m|^2}{|m|^5} \left(\frac{r_e^2}{r_i a} \right)^{2|m|} < \infty \quad \text{with } h_m := \frac{1}{2\pi} \int_0^{2\pi} h(\theta) e^{-im\theta} d\theta. \tag{11}$$

In particular, the condition (11) is satisfied for any h if $a \geq r_e^2/r_i$, but fails generically for $a < r_e^2/r_i$.

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2. PRELIMINARIES

2.1. Boundary triplets. For a linear operator B we denote $\text{dom } B$, $\ker B$, $\text{ran } B$, $\sigma(B)$ and $\rho(B)$ its domain, kernel, range, spectrum and resolvent set respectively. For a self-adjoint operator B , by $\sigma_{\text{ess}}(B)$ and $\sigma_p(B)$ we denote respectively its essential spectrum and point spectrum (i.e. the set of the eigenvalues). The scalar product in a Hilbert space \mathcal{H} will be denoted as $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ or, if there is no ambiguity, simply as $\langle \cdot, \cdot \rangle$, and it is assumed anti-linear with respect to the first argument. By $\mathcal{B}(\mathfrak{h}, \mathcal{H})$ we mean the Banach space of the bounded linear operators from a Hilbert space \mathfrak{h} to a Hilbert space \mathcal{H} , and we set $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$.

Let us recall the key points of the method of boundary triplets for self-adjoint extensions [11, 13, 16]. Our presentation mostly follows the first chapters of [11]. Let S be a closed densely defined symmetric operator in a Hilbert space \mathcal{H} . A triplet $(\mathfrak{h}, \Gamma_1, \Gamma_2)$, where \mathfrak{h} is an auxiliary Hilbert space and Γ_1 and Γ_2 are linear maps from the domain $\text{dom } S^*$ of the adjoint operator S^* to \mathfrak{h} , is called a *boundary triplet* for S if the following three conditions are satisfied:

- (a) the Green's identity $\langle u, S^*v \rangle_{\mathcal{H}} - \langle S^*u, v \rangle_{\mathcal{H}} = \langle \Gamma_1 u, \Gamma_2 v \rangle_{\mathfrak{h}} - \langle \Gamma_2 u, \Gamma_1 v \rangle_{\mathfrak{h}}$ holds for all $u, v \in \text{dom } S^*$,
- (b) the map $\text{dom } S^* \ni u \mapsto (\Gamma_1 u, \Gamma_2 u) \in \mathfrak{h} \times \mathfrak{h}$ is surjective,
- (c) $\ker \Gamma_1 \cap \ker \Gamma_2 = \text{dom } S$.

It is known that a boundary triplet for S exists if and only if S admits self-adjoint extensions, i.e. if its deficiency indices are equal, $\dim \ker(S^* - i) = \dim \ker(S^* + i) =: \mathbf{n}(S)$. A boundary triplet is not unique, but for any choice of a boundary triplet $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ for S one has $\dim \mathfrak{h} = \mathbf{n}(S)$.

Let us assume from now on that the deficiency indices of S are equal and pick a boundary triplet $(\mathfrak{h}, \Gamma_1, \Gamma_2)$, then the self-adjoint extensions of S are in a one-to-one correspondence with the self-adjoint linear relations in \mathfrak{h} (multi-valued self-adjoint operators). In the present text we prefer to keep the operator language and reformulate this result as follows, cf. [33]: Let $\Pi : \mathfrak{h} \rightarrow \text{ran } \Pi \subseteq \mathfrak{h}$ be an orthogonal projector in \mathfrak{h} and Θ be a linear operator in the Hilbert space $\text{ran } \Pi$ with the induced scalar product. Denote by $A_{\Pi, \Theta}$ the restriction of S^* to

$$\text{dom } A_{\Pi, \Theta} = \{u \in \text{dom } S^* : \Gamma_1 u \in \text{dom } \Theta \text{ and } \Pi \Gamma_2 u = \Theta \Gamma_1 u\},$$

then $A_{\Pi, \Theta}$ is symmetric/closed/self-adjoint iff Θ possesses the respective property as an operator in $\text{ran } \Pi$, and one has $\overline{A_{\Pi, \Theta}} = A_{\Pi, \overline{\Theta}}$, where as usual the bar means taking the closure. Furthermore, any self-adjoint extension of S is of the form $A_{\Pi, \Theta}$.

The spectral analysis of the self-adjoint extensions can be carried out using the associated Weyl functions. Namely, let A be the restriction of S^* to $\ker \Gamma_1$, i.e. corresponds to $(\Pi, \Theta) = (0, 0)$ in the above notation, which is a self-adjoint operator. For $z \in \rho(A)$ the restriction $\Gamma_1 : \ker(S^* - z) \rightarrow \mathfrak{h}$ is a bijection, and we denote its inverse by G_z . The map $z \mapsto G_z$, called the associated γ -field, is then a holomorphic map from $\rho(A)$ to $\mathcal{B}(\mathfrak{h}, \mathcal{H})$ with

$$G_z - G_w = (z - w)(A - z)^{-1}G_w, \quad z, w \in \rho(A). \quad (12)$$

The *Weyl function* associated with the boundary triplet is then the holomorphic map

$$\rho(A) \ni z \mapsto M_z := \Gamma_2 G_z \in \mathcal{B}(\mathfrak{h}).$$

To describe the spectral properties of the self-adjoint operators $A_{\Pi, \Theta}$ let us consider first the case $\Pi = 1$, then Θ is a self-adjoint operator in \mathcal{H} , and the following holds, see Theorems 1.29 and Theorem 3.3 in [11]:

Proposition 5. *For any $z \in \rho(A) \cap \rho(A_{1,\Theta})$ one has $0 \in \rho(\Theta - M_z)$ and the resolvent formula*

$$(A_{1,\Theta} - z)^{-1} = (A - z)^{-1} + G_z(\Theta - M_z)^{-1}G_z^* \quad (13)$$

holds. In addition, for any $z \in \rho(A)$ one has the equivalences:

- (a) $z \in \sigma(A_{1,\Theta})$ iff $0 \in \sigma(\Theta - M_z)$,
- (b) $z \in \sigma_{\text{ess}}(A_{1,\Theta})$ iff $0 \in \sigma_{\text{ess}}(\Theta - M_z)$,
- (c) $z \in \sigma_p(A_{1,\Theta})$ iff $0 \in \sigma_p(\Theta - M_z)$ with G_z being an isomorphism of the eigensubspaces.
- (d) If $f \in \mathcal{H}$, then $f \in \text{ran}(A_{1,\Theta} - z)$ iff $G_z^*f \in \text{ran}(\Theta - M_z)$. If $\Theta - M_z$ is injective, the resolvent formula (13) still holds for such f .

It seems that the point (d) was not stated explicitly in earlier references, its proof is given in Appendix A.

Now let $A_{\Pi,\Theta}$ be an arbitrary self-adjoint extension. Denote by S_Π the restriction of S^* to

$$\text{dom } S_\Pi = \{u \in \text{dom } S^* : \Gamma_1 u = \Pi \Gamma_2 u = 0\},$$

which is a closed densely defined symmetric operator whose adjoint S_Π^* is the restriction of S^* to

$$\text{dom } S_\Pi^* = \{u \in \text{dom } S^* : \Gamma_1 u \in \text{ran } \Pi\},$$

then $(\text{ran } \Pi, \Gamma_1^\Pi, \Gamma_2^\Pi)$, with $\Gamma_j^\Pi := \Pi \Gamma_j$, is a boundary triplet for S_Π , and the restriction of S_Π^* to $\ker \Gamma_1^\Pi$ is the same operator A as previously. The associated γ -field and Weyl function take the form

$$z \mapsto G_z^\Pi := G_z \Pi^*, \quad z \mapsto M_z^\Pi := \Pi M_z \Pi^*,$$

and $\text{dom } A_{\Pi,\Theta} := \{u \in \text{dom } S_\Pi^* : \Gamma_2^\Pi u = \Theta \Gamma_1^\Pi u\}$, see [11, Remark 1.30]. A direct application of Proposition 5 gives

Corollary 6. *For any $z \in \rho(A) \cap \rho(A_{\Pi,\Theta})$ one has $0 \in \rho(\Theta - M_z^\Pi)$ and the resolvent formula*

$$(A_{\Pi,\Theta} - z)^{-1} = (A - z)^{-1} + G_z^\Pi(\Theta - M_z^\Pi)^{-1}(G_z^\Pi)^* \quad (14)$$

holds, and, in addition, for any $z \in \rho(A)$ one has

- (a) $z \in \sigma(A_{\Pi,\Theta})$ iff $0 \in \sigma(\Theta - M_z^\Pi)$,
- (b) $z \in \sigma_{\text{ess}}(A_{\Pi,\Theta})$ iff $0 \in \sigma_{\text{ess}}(\Theta - M_z^\Pi)$,
- (c) $z \in \sigma_p(A_{\Pi,\Theta})$ iff $0 \in \sigma_p(\Theta - M_z^\Pi)$ with G_z^Π being an isomorphism of the eigensubspaces.
- (d) If $f \in \mathcal{H}$, then $f \in \text{ran}(A_{\Pi,\Theta} - z)$ iff $(G_z^\Pi)^*f \in \text{ran}(\Theta - M_z^\Pi)$. If $\Theta - M_z^\Pi$ is injective, the resolvent formula (14) still holds for such f .

2.2. Singular perturbations. In this section let us recall a special approach to the construction of boundary triplets as presented in [32] and [33] or in [11, Section 1.4.2]. Let A be a self-adjoint operator in a Hilbert space \mathcal{H} , then we denote by \mathcal{H}_A the Hilbert space given by the linear space $\text{dom } A$ endowed with the scalar product $\langle u, v \rangle_A = \langle u, v \rangle_{\mathcal{H}} + \langle Au, Av \rangle_{\mathcal{H}}$. Let \mathfrak{h} be an auxiliary Hilbert space and $\tau : \mathcal{H}_A \rightarrow \mathfrak{h}$ be a bounded linear map which is *surjective* and whose kernel $\ker \tau$ is *dense in* \mathcal{H} , then the restriction S of A to $\ker \tau$ is a closed densely defined symmetric operator in \mathcal{H} . To simplify the formulas we assume additionally that

$$0 \in \rho(A),$$

which always holds in the subsequent applications. For $z \in \rho(A)$ consider the maps

$$G_z := (\tau(A - \bar{z})^{-1})^* \in \mathcal{B}(\mathfrak{h}, \mathcal{H}), \quad M_z := \tau(G_z - G_0) \equiv z\tau A^{-1}G_z \in \mathcal{B}(\mathfrak{h}). \quad (15)$$

Proposition 7. *The adjoint S^* is given by*

$$\text{dom } S^* := \{u = u_0 + G_0 f_u : u_0 \in \text{dom } A \text{ and } f_u \in \mathfrak{h}\}, \quad S^* u = A u_0.$$

Furthermore, the triplet $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ with $\Gamma_1 u := f_u$ and $\Gamma_2 u := \tau u_0$ is a boundary triplet for S , and the associated γ -field G_z and Weyl function M_z are given by (15).

Example 8. Let A_\pm be self-adjoint operators in Hilbert spaces \mathcal{H}_\pm with $0 \in \rho(A_\pm)$, and let $\mathfrak{h}_\pm, \tau_\pm, S_\pm, G_z^\pm, M_z^\pm, \Gamma_j^\pm$ be the spaces and maps defined as above and associated with A_\pm . For $\nu \in \mathbb{R} \setminus \{0\}$ consider the operator $A := A_- \oplus \nu A_+$ acting in the Hilbert space $\mathcal{H} := \mathcal{H}_- \oplus \mathcal{H}_+$. Set $\tau = \tau_- \oplus \nu \tau_+$, then the restriction S of A to $\ker \tau$ has again the structure of a direct sum, $S = S_- \oplus \nu S_+$, with γ -field and Weyl function given by

$$G_z = G_z^- \oplus G_z^+, \quad M_z = M_z^- \oplus \nu M_z^+.$$

Thus, by the preceding considerations, the adjoint S^* acts on the domain

$$\text{dom } S^* = \{u = (u_-, u_+) : u_\pm = u_0^\pm + G_0^\pm \phi_\pm, \quad u_0^\pm \in \text{dom } A_\pm, \quad \phi_\pm \in \mathfrak{h}_\pm\} \quad (16)$$

by $S^*(u_-, u_+) = A(u_0^-, u_0^+)$, and one can take $(\mathfrak{h}_- \oplus \mathfrak{h}_+, \Gamma_1, \Gamma_2)$ as a boundary triplet for S ,

$$\Gamma_1 u = (\phi_-, \phi_+), \quad \Gamma_2 u = (\tau_- u_0^-, \nu \tau_+ u_0^+). \quad (17)$$

3. BOUNDARY TRIPLETS FOR INDEFINITE LAPLACIANS

We start with some constructions for the closed symmetric operator

$$S = (-\Delta_-^{\min}) \oplus \mu \Delta_+^{\min}, \quad \Delta_\pm^{\min} : L^2(\Omega_\pm) \supset H_0^2(\Omega_\pm) \rightarrow L^2(\Omega_\pm), \quad (18)$$

where

$$\begin{aligned} H_0^2(\Omega_-) &:= \{u_- \in H^2(\Omega_-) : (\gamma_0^- u_-, \gamma_1^- u_-) = (0, 0)\}, \\ H_0^2(\Omega_+) &:= \{u_+ \in H^2(\Omega_+) : (\gamma_0^+ u_+, \gamma_1^+ u_+, \gamma_0^\partial u_+, \gamma_1^\partial u_+) = (0, 0, 0, 0)\}. \end{aligned}$$

Here and later on, $H^m(\Omega_\pm)$ denotes the usual Sobolev-Hilbert space of the square-integrable functions on Ω_\pm with square integrable partial (distributional) derivatives of any order $k \leq m$, and the linear operators

$$\begin{aligned} \gamma_0^\pm : H^2(\Omega_\pm) &\rightarrow H^{\frac{3}{2}}(\Sigma), & \gamma_1^\pm : H^2(\Omega_\pm) &\rightarrow H^{\frac{1}{2}}(\Sigma), \\ \gamma_0^\partial : H^2(\Omega_+) &\rightarrow H^{\frac{3}{2}}(\partial\Omega), & \gamma_1^\partial : H^2(\Omega_+) &\rightarrow H^{\frac{1}{2}}(\partial\Omega), \end{aligned}$$

are the usual trace maps first defined on $u_\pm \in C^\infty(\overline{\Omega_\pm})$ by

$$\begin{aligned} \gamma_0^\pm u_\pm(x) &:= u_\pm(x), & \gamma_1^\pm u_\pm(x) &:= N_\pm(x) \cdot \nabla u_\pm(x), & x &\in \Sigma, \\ \gamma_0^\partial u_+(x) &:= u_+(x), & \gamma_1^\partial u_+(x) &:= N_\partial(x) \cdot \nabla u_+(x), & x &\in \partial\Omega, \end{aligned}$$

with N_∂ being the outer unit normal on $\partial\Omega$, and then extended by continuity. It is well known, see e.g. [26, Chapter 1, Section 8.2], that the maps

$$\begin{aligned} H^2(\Omega_-) \ni u_- &\mapsto (\gamma_0^- u_-, \gamma_1^- u_-) \in H^{\frac{3}{2}}(\Sigma) \oplus H^{\frac{1}{2}}(\Sigma), \\ H^2(\Omega_+) \ni u_+ &\mapsto (\gamma_0^+ u_+, \gamma_1^+ u_+, \gamma_0^\partial u_+, \gamma_1^\partial u_+) \in H^{\frac{3}{2}}(\Sigma) \oplus H^{\frac{1}{2}}(\Sigma) \oplus H^{\frac{3}{2}}(\partial\Omega) \oplus H^{\frac{1}{2}}(\partial\Omega) \end{aligned}$$

are bounded and surjective.

We remark that both Σ and $\partial\Omega$ can be made smooth compact Riemannian manifolds. For $\Xi = \Sigma$ or $\Xi = \partial\Omega$, the fractional order Sobolev-Hilbert spaces $H^s(\Xi)$ with $s \in \mathbb{R}$, are defined in the standard way as the completions of $C^\infty(\Xi)$ with respect to the scalar products

$$\langle \phi_1, \phi_2 \rangle_{H^s(\Xi)} := \langle \phi_1, (-\Delta_\Xi + 1)^s \phi_2 \rangle_{L^2(\Xi)},$$

where Δ_Ξ is the (negative) Laplace-Beltrami operator in $L^2(\Xi)$, see e.g. [26, Remark 7.6, Chapter 1, Section 7.3], and then $(-\Delta_\Xi + 1)^{\frac{s}{2}}$ extends to a unitary map from $H^r(\Xi)$ to $H^{r-s}(\Xi)$. In what follows we denote for shortness

$$\Lambda := \sqrt{-\Delta_\Sigma + 1}, \quad \Lambda_\partial := \sqrt{-\Delta_{\partial\Omega} + 1}.$$

By Green's formula, the linear operators γ_0^\pm , γ_1^\pm , γ_0^∂ and γ_1^∂ can be then extended to continuous (with respect to the graph norm) maps

$$\begin{aligned} \gamma_0^\pm : \text{dom } \Delta_\pm^{\max} &\rightarrow H^{-\frac{1}{2}}(\Sigma), & \gamma_1^\pm : \text{dom } \Delta_\pm^{\max} &\rightarrow H^{-\frac{3}{2}}(\Sigma), \\ \gamma_0^\partial : \text{dom } \Delta_+^{\max} &\rightarrow H^{-\frac{1}{2}}(\partial\Omega), & \gamma_1^\partial : \text{dom } \Delta_+^{\max} &\rightarrow H^{-\frac{3}{2}}(\partial\Omega), \end{aligned} \quad (19)$$

where $\Delta_\pm^{\max} := (\Delta_\pm^{\min})^*$ acts as the distributional Laplacian on the domain

$$\text{dom } \Delta_\pm^{\max} := \{u_\pm \in L^2(\Omega_\pm) : \Delta u_\pm \in L^2(\Omega_\pm)\},$$

see [26, Chapter 2, Section 6.5]. Now consider the operator

$$A = (-\Delta_-^D) \oplus \mu \Delta_+^D$$

acting in $L^2(\Omega) \equiv L^2(\Omega_-) \oplus L^2(\Omega_+)$, where Δ_\pm^D are the Dirichlet Laplacians in $L^2(\Omega_\pm)$, i.e.

$$\begin{aligned} \text{dom } \Delta_-^D &= \{u_- \in H^2(\Omega_-) : \gamma_0^- u_- = 0\}, \\ \text{dom } \Delta_+^D &= \{u_+ \in H^2(\Omega_+) : (\gamma_0^+ u_+, \gamma_0^\partial u_+) = (0, 0)\}. \end{aligned}$$

As both Δ_\pm^D are self-adjoint with compact resolvents, the same applies to A . The maps

$$\begin{aligned} \tau_- : \text{dom } \Delta_-^D &\rightarrow H^{\frac{1}{2}}(\Sigma), & \tau_- u_- &:= \gamma_1^- u_-, \\ \tau_+ : \text{dom } \Delta_+^D &\rightarrow H^{\frac{1}{2}}(\Sigma) \oplus H^{\frac{1}{2}}(\partial\Omega), & \tau_+ u_+ &:= (\gamma_1^+ u_+, \gamma_1^\partial u_+), \end{aligned}$$

are linear, continuous, surjective, and their kernels are dense in $L^2(\Omega_\pm)$. Moreover, Δ_\pm^{\min} is exactly the restriction of Δ_\pm^D to $\ker \tau_\pm$. Therefore, we may use the construction of Example 8 with $\nu = -\mu$ to obtain a description of the self-adjoint extensions of S from (18). To this end, an expression for the associated operators G_z^\pm and M_z^\pm is needed. These were already obtained in [33, Example 5.5], and we recall the final result. The Poisson operators

$$P_z^- : H^s(\Sigma) \rightarrow \text{dom } \Delta_-^{\max}, \quad P_z^+ : H^s(\Sigma) \oplus H^s(\partial\Omega) \rightarrow \text{dom } \Delta_+^{\max}, \quad s \geq -\frac{1}{2},$$

are defined through the solutions of the respective boundary value problems,

$$\begin{aligned} P_z^- \phi = f &\text{ iff } \begin{cases} -\Delta_-^{\max} f = z f, \\ \gamma_0^- f = \phi, \end{cases} \quad z \in \rho(-\Delta_-^D), \\ P_z^+ (\varphi, \psi) = g &\text{ iff } \begin{cases} -\Delta_+^{\max} g = z g, \\ \gamma_0^+ g = \varphi, \\ \gamma_0^\partial g = \psi. \end{cases} \quad z \in \rho(-\Delta_+^D), \end{aligned}$$

and the associated (energy-dependent) Dirichlet-to-Neumann operators are given by

$$\begin{aligned} D_z^- : H^s(\Sigma) &\rightarrow H^{s-1}(\Sigma), \quad D_z^- := \gamma_1^- P_z^-, \quad s \geq -\frac{1}{2}, \\ D_z^+ : H^s(\Sigma) \oplus H^s(\partial\Omega) &\rightarrow H^{s-1}(\Sigma) \oplus H^{s-1}(\partial\Omega), \quad s \geq -\frac{1}{2}, \\ D_z^+(\varphi, \psi) &:= \left(\gamma_1^+ P_z^+(\varphi, \psi), \gamma_1^\partial P_z^+(\varphi, \psi) \right), \end{aligned}$$

then

$$\begin{aligned} G_z^- &= -P_z^- \Lambda, & M_z^- &= (D_0^- - D_z^-) \Lambda, \\ G_z^+ &= -P_z^+ (\Lambda \oplus \Lambda_\partial), & M_z^+ &= (D_0^+ - D_z^+) (\Lambda \oplus \Lambda_\partial). \end{aligned}$$

Thus, by Remark 8, the adjoint S^* acts as $S^*u = (-\Delta_-^{\max} u_-, \mu \Delta_+^{\max} u_+)$ on the domain $\text{dom}(\Delta_-^{\max} \oplus \Delta_+^{\max})$, and using (16) and (17) one obtains the boundary triplet $(\mathfrak{h}, \Gamma_1, \Gamma_2)$ for S with $\mathfrak{h} = H^{\frac{1}{2}}(\Sigma) \oplus H^{\frac{1}{2}}(\Sigma) \oplus H^{\frac{1}{2}}(\partial\Omega)$ and

$$\Gamma_1 u = - \begin{pmatrix} \Lambda^{-1} \gamma_0^- u_- \\ \Lambda^{-1} \gamma_0^+ u_+ \\ \Lambda_\partial^{-1} \gamma_0^\partial u_+ \end{pmatrix}, \quad \Gamma_2 u = \begin{pmatrix} \gamma_1^-(u_- - P_0^- \gamma_0^- u_-) \\ -\mu \gamma_1^+(u_+ - P_0^+ (\gamma_0^+ u_+, \gamma_0^\partial u_+)) \\ -\mu \gamma_1^\partial(u_+ - P_0^+ (\gamma_0^+ u_+, \gamma_0^\partial u_+)) \end{pmatrix}.$$

The associated γ -field G_z and M_z are given by

$$G_z \begin{pmatrix} \varphi_- \\ \varphi_+ \\ \varphi_\partial \end{pmatrix} = - \begin{pmatrix} P_z^- \Lambda \varphi_- \\ P_{-\frac{z}{\mu}}^+ (\Lambda \varphi_+, \Lambda_\partial \varphi_\partial) \end{pmatrix}, \quad M_z \begin{pmatrix} \varphi_- \\ \varphi_+ \\ \varphi_\partial \end{pmatrix} = \begin{pmatrix} (D_0^- - D_z^-) \Lambda \varphi_- \\ -\mu (D_0^+ - D_{-\frac{z}{\mu}}^+) (\Lambda \varphi_+, \Lambda_\partial \varphi_\partial) \end{pmatrix}.$$

Let us represent the operator L_μ given by (3) and (4) in the form $A_{\Pi, \Theta}$. Remark first that, in view of the elliptic regularity, see e.g. [18, Proposition III.5.2], we have

$$\begin{aligned} H^2(\Omega_-) &= \{u_- \in L^2(\Omega_-) : \Delta_-^{\max} u_- \in L^2(\Omega_-), \gamma_0^+ u_- \in H^{\frac{3}{2}}(\Sigma)\} \\ H^2(\Omega_+) &= \{u_+ \in L^2(\Omega_+) : \Delta_+^{\max} u_+ \in L^2(\Omega_+), (\gamma_0^+ u_+, \gamma_0^\partial u_+) \in H^{\frac{3}{2}}(\Sigma) \oplus H^{\frac{3}{2}}(\partial\Omega)\}. \end{aligned}$$

Therefore, L_μ is exactly the restriction of S^* to the functions $u = (u_-, u_+)$ with

$$\gamma_0^- u_- = \gamma_0^+ u_+ =: \gamma_0 u, \quad \gamma_0^\partial u_+ = 0, \quad \gamma_0 u \in H^{\frac{3}{2}}(\Sigma), \quad \gamma_1^- u_- = \mu \gamma_1^+ u_+. \quad (20)$$

The first two conditions can be rewritten as $\Gamma_1 u \in \text{ran } \Pi$, where Π is the orthogonal projector in \mathfrak{h} given by

$$\Pi(\varphi_-, \varphi_+, \varphi_\partial) = \frac{1}{2}(\varphi_- + \varphi_+, \varphi_- + \varphi_+, 0).$$

For the subsequent computations it is useful to introduce the unitary operator

$$U : \text{ran } \Pi \rightarrow H^{\frac{1}{2}}(\Sigma), \quad U(\varphi, \varphi, 0) = \sqrt{2} \varphi,$$

then

$$U \Pi \Gamma_2 u = \frac{1}{\sqrt{2}} \left[\gamma_1^-(u_- - P_0^- \gamma_0^- u_-) - \mu \gamma_1^+(u_+ - P_0^+ (\gamma_0^+ u_+, \gamma_0^\partial u_+)) \right],$$

and the third and the fourth conditions in (20) can be rewritten respectively as

$$\Gamma_1 u \in U^* \text{dom } \Theta_\mu, \quad U \Pi \Gamma_2 u = \Theta_\mu U \Gamma_1 u,$$

where Θ_μ is the symmetric operator in $H^{\frac{1}{2}}(\Sigma)$ given by

$$\Theta_\mu := \frac{1}{2}(D_0^- - \mu \tilde{D}_0^+) \Lambda, \quad \text{dom } \Theta_\mu = H^{\frac{5}{2}}(\Sigma),$$

with

$$\tilde{D}_z^+ := \gamma_1^+ \tilde{P}_z^+, \quad \tilde{P}_z^+ := P_z^+(\cdot, 0).$$

Therefore, one has the representation $L_\mu = A_{\Pi, U^* \Theta_\mu U}$, and, due to the unitarity of U and to the discussion of Section 2, the operator L_μ is self-adjoint/essentially self-adjoint in $L^2(\Omega)$ if and only if Θ_μ has the respective property as an operator in $H^{\frac{1}{2}}(\Sigma)$. Remark that for the associated maps $G_z^\Pi := G_z \Pi^*$ and $M_z^\Pi := \Pi M_z \Pi^*$ (see Subsection 2.1) one has

$$G_z^\Pi U^* \varphi = -\frac{1}{\sqrt{2}} \begin{pmatrix} P_z^- \Lambda \varphi \\ \tilde{P}_{-\frac{z}{\mu}}^+ \Lambda \varphi \end{pmatrix}, \quad U M_z^\Pi U^* = \frac{1}{2} \left((D_0^- - D_z^-) - \mu (\tilde{D}_0^+ - \tilde{D}_{-\frac{z}{\mu}}^+) \right) \Lambda. \quad (21)$$

4. PROOFS OF MAIN RESULTS

With the above preparations, the proofs will be based on an application of the theory of pseudodifferential operators, see e.g. [37] and [38]. At first we recall some known results adapted to our setting.

If $\Psi \in \mathcal{B}(H^s(\Sigma), H^{s-k}(\Sigma))$ is a symmetric pseudodifferential operator of order k , we set $k_0 := \max(k, 0)$ and denote by Ψ^{\min} and Ψ^0 the symmetric operators in $L^2(\Sigma)$ given by the restriction of Ψ to $\text{dom } \Psi^{\min} = C^\infty(\Sigma)$ and $\text{dom } \Psi^0 = H^{k_0}(\Sigma)$ respectively, then $\Psi^0 \subseteq \overline{\Psi^{\min}} \subseteq \overline{\Psi^0}$. Furthermore, if Ψ is elliptic, then Ψ^0 is closed and, hence, $\overline{\Psi^{\min}} = \Psi^0$. Since $\text{dom}(\Psi^{\min})^* = \{f \in L^2(\Sigma) : \Psi f \in L^2(\Sigma)\}$, for elliptic Ψ one has $\text{dom}(\Psi^{\min})^* \subseteq H^{k_0}(\Sigma) = \text{dom } \overline{\Psi^{\min}}$, and so Ψ^{\min} is essentially self-adjoint and Ψ^0 is self-adjoint. It is important to recall that for $k = 1$ one does not need the ellipticity:

Lemma 9. *If Ψ is a symmetric first order pseudodifferential operator, then Ψ^{\min} , and then also Ψ^0 , is essentially self-adjoint in $L^2(\Sigma)$.*

Proof. By [37, Proposition 7.4], for any $f \in L^2(\Sigma)$ with $\Psi f \in L^2(\Sigma)$ there exist $(f_j) \subset C^\infty(\Sigma)$ such that $f_j \rightarrow f$ and $\Psi^{\min} f_j \rightarrow \Psi f$ in $L^2(\Sigma)$, which literally means that $\text{dom}(\Psi^{\min})^* \subseteq \text{dom } \overline{\Psi^{\min}}$. \square

In what follows, instead of studying Θ_μ in $H^{\frac{1}{2}}(\Sigma)$ we prefer to deal with the unitarily equivalent operator $\Phi_\mu := \Lambda^{\frac{1}{2}} \Theta_\mu \Lambda^{-\frac{1}{2}}$ acting in $L^2(\Sigma)$. Set

$$\Psi_\mu := \frac{1}{2} \Lambda^{\frac{1}{2}} (D_0^- - \mu \tilde{D}_0^+) \Lambda^{\frac{1}{2}},$$

then Φ_μ is the restriction of Ψ_μ to $\text{dom } \Phi_\mu = H^2(\Sigma)$. Furthermore, denote by Ψ_μ^{\min} and Ψ_μ^0 the symmetric operators in $L^2(\Sigma)$ given respectively by the restrictions of Ψ_μ to $\text{dom } \Psi_\mu^{\min} = C^\infty(\Sigma)$ and to $\text{dom } \Psi_\mu^0 = H^{k_0}(\Sigma)$, where k is the order of Ψ_μ and $k_0 = \max(k, 0)$. Remark that we always have $k \leq 2$, hence, $\Psi_\mu^{\min} \subseteq \Phi_\mu \subseteq \Psi_\mu^0$.

Proof of Theorem 1. Assume first that $\mu \neq 1$. Let us show that the operator Θ_μ is self-adjoint in $H^{\frac{1}{2}}(\Sigma)$, then this will imply the self-adjointness of L_μ in $L^2(\Omega)$. It is a classical result that D_0^\pm are first order pseudodifferential operators with the principal symbol $|\xi|$, see e.g. [38, Chapter 7, Section 11], and, in view of the definition, the same applies then to \tilde{D}_0^+ . Then Ψ_μ is a pseudodifferential operator with principal symbol $\frac{1-\mu}{2} |\xi|^2$. As such a principal symbol is non-vanishing, Ψ_μ is a second order elliptic pseudodifferential operator and, by the results recalled at the beginning of the section, $\Phi_\mu \equiv \Psi_\mu^0$ is self-adjoint on the domain $H^2(\Sigma)$. Hence, since $\Lambda^{\frac{1}{2}} : H^{\frac{1}{2}}(\Sigma) \rightarrow L^2(\Sigma)$ is unitary, the operator $\Theta_\mu = \Lambda^{-\frac{1}{2}} \Phi_\mu \Lambda^{\frac{1}{2}}$ is self-adjoint

on the initial domain $H^{\frac{5}{2}}(\Sigma)$, which implies the self-adjointness of L_μ on the initial domain $\mathcal{D}_\mu^2(\Omega \setminus \Sigma)$. Due to (2) we have $\text{dom } L_\mu \subseteq H_0^1(\Omega)$, and the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ proves that the resolvent of L_μ is a compact operator.

Let $\mu = 1$, then Ψ_1 is a first order pseudodifferential operator, and Ψ_1^{\min} is essentially self-adjoint due to Lemma 9. Then Φ_1 is also essentially self-adjoint being a symmetric extension of Ψ_1^{\min} . The unitarity of $\Lambda^{\frac{1}{2}} : H^{\frac{1}{2}}(\Sigma) \rightarrow L^2(\Sigma)$ implies the essential self-adjointness of Θ_1 in $H^{\frac{1}{2}}(\Sigma)$ and, in turn, that of L_1 in $L^2(\Omega)$. \square

Recall that in what follows we denote by \mathcal{L}_1 the unique self-adjoint extension of L_1 . In view of the discussion of Section 3 one has $\mathcal{L}_1 = A_{\Pi, U^* \Theta U}$ with $\Theta = \overline{\Theta_1}$ being the closure (and the unique self-adjoint extension) of Θ_1 in $H^{\frac{1}{2}}(\Sigma)$.

Proof of Theorem 2. Assume that $n = 2$ and $\mu = 1$, then $\Psi_1 = \frac{1}{2}\Lambda^{\frac{1}{2}}(D_0^- - \tilde{D}_0^+)\Lambda^{\frac{1}{2}}$. It is well known that the *full* symbol of the classical Dirichlet-to-Neumann map (at $z = 0$) on a smooth bounded two-dimensional domain with respect to the arclength is equal to $|\xi|$, see [14, Proposition 1] for a direct proof or [25, Section 1] for an iterative computation. It follows that Ψ_1 is a symmetric pseudodifferential operator of order $(-\infty)$, hence $\Psi_1^0 : L^2(\Sigma) \rightarrow L^2(\Sigma)$ is bounded, self-adjoint and compact. As Φ_1 is densely defined, it follows that $\overline{\Phi_1} = \Psi_1^0$. Since $\Lambda^{\frac{1}{2}} : H^{\frac{1}{2}}(\Sigma) \rightarrow L^2(\Sigma)$ is unitary, the closure of Θ_1 is given by $\Theta = \Lambda^{-\frac{1}{2}}\Phi_1\Lambda^{\frac{1}{2}} : H^{\frac{1}{2}}(\Sigma) \rightarrow H^{\frac{1}{2}}(\Sigma)$, and it is a compact self-adjoint operator in $H^{\frac{1}{2}}(\Sigma)$. As $\text{dom } \Theta = H^{\frac{1}{2}}(\Sigma)$, the boundary condition $\Gamma_1 u \in U^* \text{dom } \Theta$ takes the form $\gamma_0^- u_- = \gamma_0^+ u_+ \in H^{-\frac{1}{2}}(\Sigma)$, $\gamma_0^\partial u_+ = 0$, and, in view of (19), the domain of $\mathcal{L}_1 = A_{\Pi, U^* \Theta U}$ is given by (5).

Let us study the spectral properties of \mathcal{L}_1 using Corollary 6. As $U^* \Theta U - M_0^\Pi \equiv U^* \Theta U$ is compact, one has $0 \in \sigma_{\text{ess}}(U^* \Theta U - M_0^\Pi)$ implying $0 \in \sigma_{\text{ess}}(\mathcal{L}_1)$. To prove the reverse inclusion $\sigma_{\text{ess}}(\mathcal{L}_1) \subseteq \{0\}$ we note first that the operators $U^* \Theta U - M_z^\Pi$ are unitarily equivalent to $\Theta - U M_z^\Pi U^*$ and, hence, have the same spectra. Furthermore, the principal symbol of $D_0^\pm - D_\lambda^\pm$ is $\frac{\lambda}{2|\xi|}$ for any $\lambda \in \rho(-\Delta_\pm^D)$, see [24, Lemma 1.1]. As the principal symbol of Λ is $|\xi|$, it follows that, for any $z \in \rho(A)$, the operators $(D_0^- - D_z^-)\Lambda$ and $(\tilde{D}_0^+ - \tilde{D}_{-\frac{z}{\mu}}^+)\Lambda$ are bounded in $H^{\frac{1}{2}}(\Sigma)$ being pseudodifferential operators of order zero, and their principal symbols are $\frac{z}{2}$ and $(-\frac{z}{2\mu})$ respectively. By Eq. (21) it follows that the principal symbol of $\Theta - U M_z^\Pi U^*$ is simply $\frac{z}{2}$, and one can represent $\Theta - U M_z^\Pi U^* = \frac{z}{2} + K_z$, where K_z are compact operators depending holomorphically on $z \in \rho(A)$. As the operator A has compact resolvent, it follows by (12) that the only possible singularities of $z \mapsto K_z$ at the points of $\sigma(A)$ are simple poles with finite-dimensional residues. Therefore, the operator function $z \mapsto U^* \Theta U - M_z^\Pi := U^*(\Theta - U M_z^\Pi U^*)U$ satisfies the assumptions of the meromorphic Fredholm alternative on $\mathbb{C}_0 := \mathbb{C} \setminus \{0\}$, see [34, Theorem XIII.13], and either (a) $0 \in \sigma(U^* \Theta U - M_z^\Pi)$ for all $z \in \mathbb{C}_0 \setminus \rho(A)$, or (b) there exists a subset $B \subset \mathbb{C}_0$, without accumulation points in \mathbb{C}_0 , such that the inverse $(U^* \Theta U - M_z^\Pi)^{-1}$ exists and is bounded for $z \in \mathbb{C}_0 \setminus (B \cup \sigma(A))$ and extends to a meromorphic function in $\mathbb{C}_0 \setminus B$ such that the coefficients in the Laurent series at the points of B are finite-dimensional operators. The case (a) can be excluded: By Corollary 6 this would imply the presence of a non-empty non-real spectrum for \mathcal{L}_1 , which is not possible due to the self-adjointness. Therefore, we are in the case (b), and the resolvent formula (14) for $\mathcal{L}_1 \equiv A_{\Pi, U^* \Theta U}$ implies that the set $\mathbb{C}_0 \cap \sigma(\mathcal{L}_1) \cap \rho(A) \subseteq B$ has no accumulation points in \mathbb{C}_0 , and each point of this set is a discrete eigenvalue of \mathcal{L}_1 . Furthermore, by (12) the maps $z \mapsto G_z^\Pi$ can have at most simple poles with finite-dimensional residues at the points of $\sigma(A)$, and it is seen again from the resolvent formula (14) that the

only possible singularities of $z \mapsto (\mathcal{L}_1 - z)^{-1}$ at the points of $\sigma(A)$ are poles with finite-dimensional residues. It follows that each point of $\sigma(A)$ is either not in the spectrum of \mathcal{L}_1 or is its discrete eigenvalue. Therefore, \mathcal{L}_1 has no essential spectrum in $\mathbb{C} \setminus \{0\}$, and the only possible accumulation points for the discrete eigenvalues are 0 and ∞ . \square

Proof of Theorem 3. Assume $n \geq 3$ and $\mu = 1$, then again $\Psi_1 = \frac{1}{2}\Lambda^{\frac{1}{2}}(D_0^- - \widetilde{D}_0^+)\Lambda^{\frac{1}{2}}$. By [38, Chapter 12, Proposition C.1], there holds

$$D_0^- = (-\Delta_\Sigma)^{\frac{1}{2}} + B^- + C^-, \quad \widetilde{D}_0^+ = (-\Delta_\Sigma)^{\frac{1}{2}} + B^+ + C^+,$$

where C^\pm are pseudodifferential operators of order (-1) and B^\pm are pseudodifferential operator of order 0 whose principal symbols are $\pm b_0(x, \xi)$, with

$$b_0(x, \xi) = \frac{1}{2} \left(\operatorname{tr} W_x - \frac{\langle \xi, W_x^* \xi \rangle_{T_x^* \Sigma}}{\langle \xi, \xi \rangle_{T_x^* \Sigma}} \right)$$

and $W_x := dN_-(x) : T_x \Sigma \rightarrow T_x \Sigma$ being the Weingarten map and W_x^* its adjoint. Therefore, Ψ_1 is a pseudodifferential operator of order 1 whose principal symbol is $\frac{1}{2}b_0(x, \xi)|\xi|$. As already seen, Ψ_1^{\min} is then essentially self-adjoint by Lemma 9, and, as before, L_1 is essentially self-adjoint and its self-adjoint closure is $A_{\Pi, U^* \Theta U}$, where $\Theta := \overline{\Theta}_1$. As Θ_1 is a first order operator, one has $H^{\frac{3}{2}}(\Sigma) \subseteq \operatorname{dom} \Theta$. In particular, the boundary condition $\Gamma_1 u \in U^* H^{\frac{3}{2}}(\Sigma)$ entails $\gamma_0^- u_- = \gamma_0^+ u_+ \in H^{\frac{1}{2}}(\Sigma)$ and $\gamma_0^\partial u_+ = 0$. Due to the elliptic regularity, see e.g. [26, Chapter 2, Section 7.3], this can be rewritten as $u \in H_0^1(\Omega)$ and gives the inclusion (6).

(a) Recall that the principal curvatures $k_1(x), \dots, k_{n-1}(x)$ of Σ at a point x are the eigenvalues of W_x , hence,

$$\frac{1}{2} \left(k_1(x) + \dots + k_{n-1}(x) - \max_j k_j(x) \right) \leq b_0(x, \xi) \leq \frac{1}{2} \left(k_1(x) + \dots + k_{n-1}(x) - \min_j k_j(x) \right).$$

Let Σ' be a maximal connected component of Σ . If all k_j are either all strictly positive or all strictly negative on Σ' , one can estimate $a_1 \leq |b_0(x, \xi)| \leq a_2$ for all $x \in \Sigma'$ with some $a_1 > 0$ and $a_2 > 0$. Therefore, in this case Ψ_1 is a first order elliptic pseudodifferential operator and so, by the results recalled at the beginning of this section, Ψ_1^0 is self-adjoint. This implies that $\operatorname{dom} \Theta \equiv \operatorname{dom} \overline{\Theta}_1 = H^{\frac{3}{2}}(\Sigma)$. As before, the boundary condition $\Gamma_1 u \in U^* \operatorname{dom} \Theta$ for $u \in \operatorname{dom} \mathcal{L}_1$ entails $u \in H_0^1(\Omega)$, and one arrives at the equality (7). The inclusion $\operatorname{dom} \mathcal{L}_1 \subseteq H_0^1(\Omega)$ and the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ imply that \mathcal{L}_1 has compact resolvent.

(b) As $M_0^\Pi = 0$, by Corollary 6(b) and by the unitarity of U , to get $0 \in \sigma_{\text{ess}}(\mathcal{L}_1)$ it suffices to show that $0 \in \sigma_{\text{ess}}(\Theta)$. As $\Lambda^{\frac{1}{2}} : H^{\frac{1}{2}}(\Sigma) \rightarrow L^2(\Sigma)$ is a unitary operator, it is sufficient to show $0 \in \sigma_{\text{ess}}(\overline{\Phi}_1)$ for the unitarily equivalent operator $\overline{\Phi}_1 \equiv \Lambda^{\frac{1}{2}} \Theta \Lambda^{-\frac{1}{2}}$ in $L^2(\Sigma)$, which will be done by constructing a singular Weyl sequence, i.e. a sequence of non-zero functions $(u_j) \subset \operatorname{dom} \overline{\Phi}_1$ weakly converging to 0 in $L^2(\Sigma)$ and such that the ratio $\|\Psi_1 u_j\|_{L^2(\Sigma)} / \|u_j\|_{L^2(\Sigma)}$ tends to 0. While the domain of $\overline{\Phi}_1$ is not known explicitly, we know already that it contains $H^1(\Sigma)$.

Without loss of generality we assume that $\Sigma_\varepsilon := \{(x', 0) : x' \in B_\varepsilon\} \subset \Sigma$, where B_ε is the ball in \mathbb{R}^{n-1} centered at the origin and of radius $\varepsilon > 0$. The iterative procedure of [25, Section 1] shows that the full symbols of D_-^0 and \widetilde{D}_0^+ on Σ_ε in the local coordinates x' are equal to $|\xi|$, and it follows that the full symbol of Ψ_1 vanishes on Σ_ε . Hence, there exists a smoothing operator K and $\delta \in (0, \varepsilon)$ such that $\Psi_1 \tilde{u} = K \tilde{u}$ for all $u \in C_c^\infty(B_\delta)$, where \tilde{u} is the extension of u by zero to the whole of Σ . Take an orthonormal sequence $(u_j) \subset L^2(B_\delta)$

with $u_j \in C_c^\infty(B_\delta)$, then the sequence $(\tilde{u}_j) \subset H^1(\Sigma)$ is orthonormal and weakly converging to 0 in $L^2(\Sigma)$. Due to the compactness of K in $L^2(\Sigma)$ there exists a subsequence $(\Psi_1 \tilde{u}_{j_k})$ strongly converging to zero in $L^2(\Sigma)$. Therefore, the sequence $v_k := \tilde{u}_{j_k}$ is a sought singular Weyl sequence for $\overline{\Phi_1}$, which gives the result.

Suppose now $\text{dom } \mathcal{L}_1 = \mathcal{D}_1^s(\Omega \setminus \Sigma) \subseteq H^s(\Omega_-) \oplus H^s(\Omega_+)$ for some $s > 0$. As the set on the right-hand side is compactly embedded in $L^2(\Omega)$, see e.g. [1, Theorem 14.3.1], this implies the compactness of the resolvent of \mathcal{L}_1 and the equality $\sigma_{\text{ess}}(\mathcal{L}_1) = \emptyset$, which contradicts the previously proved relation $0 \in \sigma_{\text{ess}}(\mathcal{L}_1)$. \square

Remark 10. After some simple cancellations, the resolvent formula of Corollary 6 for \mathcal{L}_μ takes the following form:

$$(\mathcal{L}_\mu - z)^{-1} \begin{pmatrix} u_- \\ u_+ \end{pmatrix} = \begin{pmatrix} (-\Delta_-^D - z)^{-1} u_- \\ (\mu \Delta_+^D - z)^{-1} u_+ \end{pmatrix} - \begin{pmatrix} R_z^-(u_-, u_+) \\ R_z^+(u_-, u_+) \end{pmatrix},$$

with

$$\begin{aligned} R_z^-(u_-, u_+) &= P_z^- \left(D_z^- - \mu \tilde{D}_{-\frac{z}{\mu}}^+ \right)^{-1} \left(\gamma_1^- (-\Delta_-^D - z)^{-1} u_- - \mu \gamma_1^+ (\mu \Delta_+^D - z)^{-1} u_+ \right), \\ R_z^+(u_-, u_+) &= P_{-\frac{z}{\mu}}^+ \left(\left(D_z^- - \mu \tilde{D}_{-\frac{z}{\mu}}^+ \right)^{-1} \left(\gamma_1^- (-\Delta_-^D - z)^{-1} u_- - \mu \gamma_1^+ (\mu \Delta_+^D - z)^{-1} u_+ \right) \right). \end{aligned}$$

5. PROOF OF PROPOSITION 4

We continue using the conventions and notation introduced just before Theorem 4. In addition to (9) we have

$$\Omega_+ = B_{r_i} \cup B_{r_e, R}, \quad \Sigma = S_{r_i} \cup S_{r_e}, \quad \partial\Omega = S_R,$$

and for the subsequent computations we use the identification $L^2(S_\rho) \simeq L^2((0, 2\pi), \rho d\theta)$, then $L^2(\Sigma) \simeq L^2((0, 2\pi), r_i d\theta) \oplus L^2((0, 2\pi), r_e d\theta)$, and similar identifications hold for the Sobolev spaces.

In view of Corollary 6 and of the expressions (21), the injectivity of \mathcal{L}_1 is equivalent to the injectivity of the map

$$\mathcal{D} := D_0^- - \tilde{D}_0^+ : H^{-\frac{1}{2}}(\Sigma) \rightarrow H^{\frac{1}{2}}(\Sigma),$$

and then the condition $g = (0, g_+) \in \text{ran } \mathcal{L}_1$ is equivalent to $(G_0^{\text{II}})^* g \equiv -\gamma_1^+ (-\Delta_+^D)^{-1} g_+ \in \text{ran } \mathcal{D}$, or, as $(\Delta_+^D)^{-1} : L^2(\Omega_+) \rightarrow H^2(\Omega)$ and $\gamma_1^+ : H^2(\Omega) \rightarrow H^{\frac{1}{2}}(\Sigma)$, to

$$\mathcal{D}^{-1} \gamma_1^+ (-\Delta_+^D)^{-1} g_+ \in H^{-\frac{1}{2}}(\Sigma). \quad (22)$$

The condition will be checked using an explicit computation of the Dirichlet-to-Neumann maps D_0^\pm and of the inverse of Δ_+^D .

For a function f defined in Ω_\pm , define its Fourier coefficients with respect to the polar angle by

$$f_m(r) := \frac{1}{2\pi} \int_0^{2\pi} f(r \cos \theta, r \sin \theta) e^{-im\theta} d\theta, \quad m \in \mathbb{Z},$$

then f is reconstructed by $f(r \cos \theta, r \sin \theta) = \sum_{m \in \mathbb{Z}} f_m(r) e^{im\theta}$. Furthermore, the separation of variables shows that a function f is harmonic iff f_m satisfy the Euler equations

$$f_m''(r) + r^{-1} f_m'(r) - m^2 r^{-2} f_m(r) = 0,$$

whose linearly independent solutions are 1 and $\ln r$ for $m = 0$ and $r^{\pm m}$ for $m \neq 0$. This shows that for

$$(\phi_i, \phi_e) \in H^s(\Sigma), \quad \phi_{i/e}(\theta) = \sum_{m \in \mathbb{Z}} \phi_{i/e,m} e^{im\theta}, \quad \phi_{i/e,m} := \frac{1}{2\pi} \int_0^{2\pi} \phi_{i/e}(\theta) e^{-im\theta} d\theta,$$

one has the following expressions for the Poisson operators:

$$\begin{aligned} P_0^- \begin{pmatrix} \phi_i \\ \phi_e \end{pmatrix} (r \cos \theta, r \sin \theta) &= \frac{\ln \frac{r_e}{r}}{\ln \frac{r_e}{r_i}} \phi_{i,0} + \frac{\ln \frac{r}{r_i}}{\ln \frac{r_e}{r_i}} \phi_{e,0} \\ &+ \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{\left[\left(\frac{r_e}{r} \right)^{|m|} - \left(\frac{r}{r_e} \right)^{|m|} \right] \phi_{i,m} + \left[\left(\frac{r}{r_i} \right)^{|m|} - \left(\frac{r_i}{r} \right)^{|m|} \right] \phi_{e,m}}{\left(\frac{r_e}{r_i} \right)^{|m|} - \left(\frac{r_i}{r_e} \right)^{|m|}} e^{im\theta}, \\ &\quad (r, \theta) \in (r_i, r_e) \times (0, 2\pi), \end{aligned} \quad (23)$$

and

$$\begin{aligned} \tilde{P}_0^+ \begin{pmatrix} \phi_i \\ \phi_e \end{pmatrix} (r \cos \theta, r \sin \theta) &= \begin{cases} \sum_{m \in \mathbb{Z}} \left(\frac{r}{r_i} \right)^{|m|} \phi_{i,m} e^{im\theta}, & (r, \theta) \in (0, r_i) \times (0, 2\pi), \\ \frac{\ln \frac{R}{r}}{\ln \frac{R}{r_e}} \phi_{e,0} + \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{\left(\frac{R}{r} \right)^{|m|} - \left(\frac{r}{R} \right)^{|m|}}{\left(\frac{R}{r_e} \right)^{|m|} - \left(\frac{r_e}{R} \right)^{|m|}} \phi_{e,m} e^{im\theta}, & (r, \theta) \in (r_e, R) \times (0, 2\pi). \end{cases} \end{aligned} \quad (24)$$

It follows that

$$\begin{aligned} D_0^- \begin{pmatrix} \phi_i \\ \phi_e \end{pmatrix} &= \sum_{m \in \mathbb{Z}} B_m \begin{pmatrix} \phi_{i,m} \\ \phi_{e,m} \end{pmatrix} e^{im\theta}, \quad \tilde{D}_0^+ \begin{pmatrix} \phi_i \\ \phi_e \end{pmatrix} = \sum_{m \in \mathbb{Z}} C_m \begin{pmatrix} \phi_{i,m} \\ \phi_{e,m} \end{pmatrix} e^{im\theta}, \\ \mathcal{D} \begin{pmatrix} \phi_i \\ \phi_e \end{pmatrix} &= \sum_{m \in \mathbb{Z}} D_m \begin{pmatrix} \phi_{i,m} \\ \phi_{e,m} \end{pmatrix} e^{im\theta}, \quad D_m := B_m - C_m, \end{aligned}$$

with

$$B_0 = \begin{pmatrix} \frac{1}{r_i} \frac{1}{\ln \frac{r_e}{r_i}} & -\frac{1}{r_i} \frac{1}{\ln \frac{r_e}{r_i}} \\ -\frac{1}{r_e} \frac{1}{\ln \frac{r_e}{r_i}} & \frac{1}{r_e} \frac{1}{\ln \frac{r_e}{r_i}} \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{r_e} \frac{1}{\ln \frac{R}{r_e}} \end{pmatrix},$$

$$B_m = |m| \begin{pmatrix} \frac{1}{r_i} \frac{\left(\frac{r_e}{r_i}\right)^{|m|} + \left(\frac{r_i}{r_e}\right)^{|m|}}{\left(\frac{r_e}{r_i}\right)^{|m|} - \left(\frac{r_i}{r_e}\right)^{|m|}} & -\frac{2}{r_i} \frac{1}{\left(\frac{r_e}{r_i}\right)^{|m|} - \left(\frac{r_i}{r_e}\right)^{|m|}} \\ -\frac{2}{r_e} \frac{1}{\left(\frac{r_e}{r_i}\right)^{|m|} - \left(\frac{r_i}{r_e}\right)^{|m|}} & \frac{1}{r_e} \frac{\left(\frac{r_e}{r_i}\right)^{|m|} + \left(\frac{r_i}{r_e}\right)^{|m|}}{\left(\frac{r_e}{r_i}\right)^{|m|} - \left(\frac{r_i}{r_e}\right)^{|m|}} \end{pmatrix}, \quad m \neq 0,$$

$$C_m = |m| \begin{pmatrix} \frac{1}{r_i} & 0 \\ 0 & \frac{1}{r_e} \frac{\left(\frac{R}{r_e}\right)^{|m|} + \left(\frac{r_e}{R}\right)^{|m|}}{\left(\frac{R}{r_e}\right)^{|m|} - \left(\frac{r_e}{R}\right)^{|m|}} \end{pmatrix}, \quad m \neq 0.$$

Therefore,

$$D_0 = \begin{pmatrix} \frac{1}{r_i} \frac{1}{\ln \frac{r_e}{r_i}} & -\frac{1}{r_i} \frac{1}{\ln \frac{r_e}{r_i}} \\ -\frac{1}{r_e} \frac{1}{\ln \frac{r_e}{r_i}} & \frac{1}{r_e} \left(\frac{1}{\ln \frac{r_e}{r_i}} - \frac{1}{\ln \frac{R}{r_e}} \right) \end{pmatrix},$$

$$D_m = 2|m| \begin{pmatrix} \frac{1}{r_i} \frac{1}{\left(\frac{r_e}{r_i}\right)^{2|m|} - 1} & -\frac{1}{r_i} \frac{\left(\frac{r_e}{r_i}\right)^{|m|}}{\left(\frac{r_e}{r_i}\right)^{2|m|} - 1} \\ -\frac{1}{r_e} \frac{\left(\frac{r_e}{r_i}\right)^{|m|}}{\left(\frac{r_e}{r_i}\right)^{2|m|} - 1} & \frac{1}{r_e} \left(\frac{1}{\left(\frac{r_e}{r_i}\right)^{2|m|} - 1} - \frac{1}{\left(\frac{R}{r_e}\right)^{2|m|} - 1} \right) \end{pmatrix}, \quad m \neq 0,$$

hence, all D_m are invertible, and then \mathcal{D} is injective with the inverse

$$\mathcal{D}^{-1} \begin{pmatrix} \phi_i \\ \phi_e \end{pmatrix} = \sum_{m \in \mathbb{Z}} D_m^{-1} \begin{pmatrix} \phi_{i,m} \\ \phi_{e,m} \end{pmatrix} e^{im\theta}, \quad (25)$$

which shows the injectivity of \mathcal{L}_1 . Furthermore, for $m \neq 0$ we have

$$D_m^{-1} = -\frac{1}{2|m|} \begin{pmatrix} r_i \left(1 - \left(\frac{r_e^2}{r_i R}\right)^{2|m|}\right) & r_e \left(\frac{r_e}{r_i}\right)^{|m|} \left(1 - \left(\frac{r_e}{R}\right)^{2|m|}\right) \\ r_i \left(\frac{r_e}{r_i}\right)^{|m|} \left(1 - \left(\frac{r_e}{R}\right)^{2|m|}\right) & r_e \left(1 - \left(\frac{r_e}{R}\right)^{2|m|}\right) \end{pmatrix},$$

and we conclude that a function $(\phi_i, \phi_e) \in H^{\frac{1}{2}}(\Sigma)$ belongs to $\text{ran } \mathcal{D}$ iff $\mathcal{D}^{-1}(\phi_i, \phi_e) \in H^{-\frac{1}{2}}(\Sigma)$, i.e. iff

$$\sum_{m \neq 0} \frac{1}{|m|} \left\| D_m^{-1} \begin{pmatrix} \phi_{i,m} \\ \phi_{e,m} \end{pmatrix} \right\|_{\mathbb{C}^2}^2 < \infty. \quad (26)$$

Therefore, the condition (22) is equivalent to (26) for

$$(\phi_i, \phi_e) := \gamma_1^+ f, \quad f := (\Delta_+^D)^{-1} g_+. \quad (27)$$

Remark first that f vanishes in B_{r_i} , hence, $\phi_i = 0$ and $f_m(r) = 0$ for $r < r_i$ and $m \in \mathbb{Z}$. To study the problem in $B_{r_e, R}$, let us pass to the Fourier coefficients, then we arrive to the system of equations

$$f_m''(r) + r^{-1} f_m'(r) - m^2 r^{-2} f_m(r) = h_m 1_{(a,b)}(r), \quad r_e < r < R, \quad f_m(r_e) = f_m(R) = 0, \quad (28)$$

and we have

$$\begin{pmatrix} \phi_i \\ \phi_e \end{pmatrix} = - \sum_{m \in \mathbb{Z}} \begin{pmatrix} 0 \\ f_m'(r_e) \end{pmatrix} e^{im\theta}. \quad (29)$$

One solves (28) using the variation of constants, and for $m \neq 0$ the solutions are

$$\begin{aligned} f_m(r) &= \alpha_m r^m + \beta_m r^{-m} + \frac{h_m r^m}{2m} \int_{r_e}^r s^{1-m} 1_{(a,b)}(s) ds - \frac{h_m r^{-m}}{2m} \int_{r_e}^r s^{1+m} 1_{(a,b)}(s) ds, \\ \alpha_m &= -\frac{h_m}{2m r_e^m} \frac{1}{\left(\frac{R}{r_e}\right)^m - \left(\frac{r_e}{R}\right)^m} \int_a^b \left(\left(\frac{R}{s}\right)^m - \left(\frac{s}{R}\right)^m \right) s ds, \quad \beta_m = -r_e^{2m} \alpha_m, \end{aligned}$$

and

$$f_m'(r_e) = \frac{m h_m}{r_e} (\alpha_m r_e^m - \beta_m r_e^{-m}) = -\frac{1}{r_e} \frac{\int_a^b \left(\left(\frac{R}{s}\right)^{|m|} - \left(\frac{s}{R}\right)^{|m|} \right) s ds}{\left(\frac{R}{r_e}\right)^{|m|} - \left(\frac{r_e}{R}\right)^{|m|}}.$$

Then for large m one has

$$f_m'(r_e) = -\frac{(a^2 + o(1)) h_m}{r_e |m|} \left(\frac{r_e}{a}\right)^{|m|}, \quad D_m^{-1} \begin{pmatrix} 0 \\ -f_m'(r_e) \end{pmatrix} = h_m \begin{pmatrix} \frac{a^2 + o(1)}{2m^2} \left(\frac{r_e^2}{r_i a}\right)^{|m|} \\ \frac{a^2 + o(1)}{2m^2} \left(\frac{r_e}{a}\right)^{|m|} \end{pmatrix},$$

and the condition (26) for the function (27) takes the form (11), which finishes the proof.

One should remark that the condition (11) can still hold for $a < r_e^2/r_i$ if the Fourier coefficients h_m of h are very fast decaying for large m , i.e. if h extends to an analytic function in a suitable complex neighborhood of the unit circle.

Remark 11. At last we note that, in view of the injectivity of \mathcal{L}_1 , the expression for its inverse given in Remark 10 can be extended naturally to a linear map $\mathcal{L}^{-1} : L^2(\Omega) \rightarrow \mathcal{D}'(\Omega)$. As $\text{ran}(\Delta_\pm^D)^{-1} = H^2(\Omega_\pm) \cap H_0^1(\Omega_\pm)$, the finiteness of the norms $\|\mathcal{L}_1^{-1} g\|_{H^s(V)}$, $V \subseteq \Omega$, $0 \leq s \leq 1$, is equivalent to the finiteness of $\|v\|_{H^s(V)}$ for $v := (R_0^- g, R_0^+ g)$. The direct substitution of the values of (25) and (29) into (23) and (24) shows that one always has $v \in H^1(B_{r_e, R})$, while the condition $v \in L^2(B_{r_i, r_e})$ appears to be equivalent to (11), so it holds for any h for $a \geq r_e^2/r_i$ as before, otherwise a very strong regularity of h is required.

APPENDIX A. PROOF OF PROPOSITION 5(D)

Let $f \in \text{ran}(A_{1,\Theta} - z)$, then there is $g \in \text{dom } A_{1,\Theta}$ with $f = (A_{1,\Theta} - z)g$. By [11, Theorem 1.23(a)], one can uniquely represent

$$g = g_z + G_z h \quad (30)$$

with $g_z \in \text{dom } A$ and $h \in \mathfrak{h}$, and $f = (S^* - z)g = (A - z)g_z$. As $g_z \in \text{dom } A = \ker \Gamma_1$, we have $\Gamma_1 g = \Gamma_1 G_z h = h$ and $\Gamma_2 g = \Gamma_2 g_z + \Gamma_2 G_z h$. By [11, Theorem 1.23(2d)] there holds $\Gamma_2 g_z = \Gamma_2 (A - z)^{-1} f = G_z^* f$, and by definition we have $\Gamma_2 G_z h = M_z h$. Therefore, the boundary condition $\Gamma_2 g = \Theta \Gamma_1 g$ writes as $G_z^* f = (\Theta - M_z)h$ implying $G_z^* f \in \text{ran}(\Theta - M_z)$. If $\Theta - M_z$ is injective, then $A_{1,\Theta}$ is also injective by Proposition 5(c), $h = (\Theta - M_z)^{-1} G_z^* f$, and the substitution into (30) gives the relation

$$(A_{1,\Theta} - z)^{-1} f = g = g_z + G_z h = (A - z)^{-1} f + G_z (\Theta - M_z)^{-1} G_z^* f.$$

Now let $f \in \mathcal{H}$ such that $G_z^* f \in \text{ran}(\Theta - M_z)$. Take $h \in \mathfrak{h}$ with $G_z^* f = (\Theta - M_z)h$ and consider the function $g = g_z + G_z h$ with $g_z = (A - z)^{-1} f \in \text{dom } A$. By [11, Theorem 1.23(2d)] we have $g \in \text{dom } S^*$. As previously, $\Gamma_1 g = h$ and $\Gamma_2 g = G_z^* f + M_z h = (\Theta - M_z)h + M_z h = \Theta h = \Theta \Gamma_1 g$, i.e. $g \in \text{dom } A_{1,\Theta}$, and we have $(A_{1,\Theta} - z)g = (S^* - z)g = (S^* - z)(A - z)^{-1} f = f$, i.e. $f \in \text{ran}(A_{1,\Theta} - z)$.

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